

INCLUSION OF $\Lambda BV^{(p)}$ SPACES IN THE CLASSES H_ω^q

MAHDI HORMOZI

ABSTRACT. A characterization of the inclusion of Waterman-Shiba classes into classes of functions with given integral modulus of continuity is given. This corrects and extends an earlier result of a paper from 2005.

1. PRELIMINARIES

Let $\Lambda = (\lambda_i)$ be a nondecreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_i} = +\infty$ and let p be a number greater than or equal to 1. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be of bounded p - Λ -variation on a not necessarily closed subinterval $P \subset [a, b]$ if

$$V(f) := \sup \left(\sum_{i=1}^n \frac{|f(I_i)|^p}{\lambda_i} \right)^{\frac{1}{p}} < +\infty,$$

where the supremum is taken over all finite families $\{I_i\}_{i=1}^n$ of nonoverlapping subintervals of P and where $f(I_i) := f(\sup I_i) - f(\inf I_i)$ is the change of the function f over the interval I_i . The symbol $\Lambda BV^{(p)}$ denotes the linear space of all functions of bounded p - Λ -variation with domain $[0, 1]$. The Waterman-Shiba class $\Lambda BV^{(p)}$ was introduced in 1980 by M. Shiba in [9]. When $p = 1$, $\Lambda BV^{(p)}$ is the well-known Waterman class ΛBV . Some of the basic properties of functions of class $\Lambda BV^{(p)}$ were discussed by R. G. Vyas in [12] recently. More results concerned with the Waterman-Shiba classes and their applications can be found in [1], [2], [4], [6], [7], [8], [10] and [11]. $\Lambda BV^{(p)}$ equipped with the norm $\|f\|_{\Lambda, p} := |f(0)| + V(f)$ is a Banach space.

Functions in a Waterman-Shiba class $\Lambda BV^{(p)}$ are regulated [11, Thm. 2], hence integrable, and thus it makes sense to consider their integral modulus of continuity

$$\omega_q(\delta, f) := \sup_{0 \leq \gamma \leq \delta} \left(\int_0^{1-\gamma} |f(t+\gamma) - f(t)|^q dt \right)^{\frac{1}{q}},$$

Date: October 4, 2012.

2000 Mathematics Subject Classification. Primary 26A15; Secondary 26A45.

Key words and phrases. generalized bounded variation, modulus of variation, Banach space.

for $0 \leq \delta \leq 1$. However, if f is defined on \mathbb{R} instead of on $[0, 1]$ and if f is 1-periodic, it is convenient to modify the definition and put

$$\omega_q(\delta, f) := \sup_{0 \leq \gamma \leq \delta} \left(\int_0^1 |f(t + \gamma) - f(t)|^q dt \right)^{\frac{1}{q}},$$

since the difference between the two definitions is then nonessential in all applications of the concept. A function $\omega : [0, 1] \rightarrow \mathbb{R}$ is said to be a modulus of continuity if it is nondecreasing, continuous and $\omega(0) = 0$. If ω is a modulus of continuity, then H_ω^q denotes the class of functions $f \in L^q[0, 1]$ for which $\omega_q(\delta, f) = O(\omega(\delta))$ as $\delta \rightarrow 0+$.

In [4], a necessary and sufficient condition for the inclusion $\Lambda BV^{(p)}$ in H_ω^1 , is given. Also, Wang [13] by using an interesting method found a necessary and sufficient condition for the embedding $H_\omega^q \subset \Lambda BV$. Here, we give a necessary and sufficient condition for the inclusion of $\Lambda BV^{(p)}$ in H_ω^q .

2. MAIN RESULT

In [3], it was claimed that the following is true.

Theorem 2.1. *For $q \in [1, \infty)$, the inclusion $\Lambda BV \subset H_q^\omega$ holds if and only if*

$$(1) \quad \limsup_{n \rightarrow \infty} \frac{1}{\omega(1/n)n^{\frac{1}{q}}} \max_{1 \leq k \leq n} \frac{k^{\frac{1}{q}}}{\left(\sum_{i=1}^k \frac{1}{\lambda_i}\right)} < +\infty.$$

The proof of sufficiency of the condition came up immediately from [5] and so, the main part of [3] concerns the proof of necessity. The theorem itself is correct but, unfortunately, there is a major mistake regarding the existence of some subsequences, which ensures that the proof of [3] is incorrect. To understand this, take $\omega(x) = x^{1/p}$. Then we can choose $\gamma_k = 2^k$, $\gamma'_k = n'_k = k$. If we take $s'_k = k$, then the condition for case (a) in [3] is satisfied, since $m(x) \leq 2^x$ by definition. But for subsequences r_k of s'_k , relation (3) in [3], $\omega(\frac{1}{2^{r_k}}) \cdot 2^{r_k/p} \leq 4^{-k}$, is not true.

Our main result provides a characterization of the embedding of a generalization of ΛBV , Waterman-Shiba classes, into classes of functions with given integral modulus of continuity. Thus, by considering $p = 1$, the correctness of [3, Thm. 1] can be verified.

Theorem 2.2. *For $p, q \in [1, \infty)$, the inclusion $\Lambda BV^{(p)} \subset H_q^\omega$ holds if and only if*

$$(2) \quad \limsup_{n \rightarrow \infty} \left\{ \frac{1}{\omega(1/n)n^{\frac{1}{q}}} \max_{1 \leq k \leq n} \frac{k^{\frac{1}{q}}}{\left(\sum_{i=1}^k \frac{1}{\lambda_i}\right)^{\frac{1}{p}}} \right\} < +\infty.$$

Proof. To observe that equation (2) is a **sufficiency** condition for the inclusion $\Lambda BV^{(p)} \subset H_\omega^q$, we prove an inequality which gives us the sufficiency :

$$\omega\left(\frac{1}{n}, f\right)_q \leq V(f) \left\{ \frac{1}{n} \max_{1 \leq k \leq n} \frac{k}{\left(\sum_{i=1}^k 1/\lambda_i\right)^{\frac{q}{p}}} \right\}^{\frac{1}{q}}.$$

Kuprikov [5] obtained Lemma 2.3 and Corollary 2.4 where $q \geq 1$.

Lemma 2.3. *Let $q \geq 1$ and suppose $F(x) = \sum_{i=1}^n x_i^q$ takes its maximum value under the following conditions*

$$\left(\sum_{i=1}^n \frac{x_i}{\lambda_i}\right) \leq 1,$$

$$x_1 \geq x_2 \geq x_3 \geq \dots \geq x_n \geq 0,$$

then, $x = (x_1, x_2, \dots, x_n)$ satisfy

$$(3) \quad x_1 = x_2 = \dots = x_k = \frac{1}{\sum_{j=1}^k 1/\lambda_j} > x_{k+1} = x_{k+2} = \dots = x_n = 0,$$

for some k ($1 \leq k \leq n$).

Corollary 2.4. *The maximum value of $F(x)$, under the conditions of Lemma 2.3, is $\max_{1 \leq k \leq n} \frac{k}{\left(\sum_{j=1}^k \frac{1}{\lambda_j}\right)^q}$.*

Lemma 2.5. *Suppose $0 < q < 1$ and the conditions of Lemma 2.3 hold, then*

$$\sum_{i=1}^n x_i^q = \frac{n}{\left(\sum_{j=1}^n \frac{1}{\lambda_j}\right)^q} = \max_{1 \leq k \leq n} \frac{k}{\left(\sum_{j=1}^k \frac{1}{\lambda_j}\right)^q}.$$

Proof. Hölder inequality yields

$$\sum_{i=1}^n x_i^q \leq n^{1-q} \sum_{i=1}^n x_i \leq \frac{n}{\left(\sum_{j=1}^n \frac{1}{\lambda_j}\right)^q}.$$

Thus $F(x)$ takes its maximum when $x_i = \frac{1}{\sum_{j=1}^n \frac{1}{\lambda_j}}$ for $1 \leq j \leq n$. \square

Now, we return to the proof of inequality:

$$\begin{aligned} \omega_q\left(\frac{1}{n}, f\right)^q &= \sup_{0 < h \leq \frac{1}{n}} \int_0^1 |f(x+h) - f(x)|^q dx \\ &= \sup_{0 < h \leq \frac{1}{n}} \int_0^{\frac{1}{n}} \sum_{k=1}^n |f(x + \frac{k-1}{n} + h) - f(x + \frac{k-1}{n})|^q dx. \end{aligned}$$

For $h \leq \frac{1}{n}$ and fixed x , denote $x_k := |f(x + \frac{k-1}{n} + h) - f(x + \frac{k-1}{n})|^p$. We reorder x_k such that

$$x_1 \geq x_2 \geq \dots \geq x_n \geq 0, \quad \left(\sum_{k=1}^n \frac{x_k}{\lambda_k} \right)^{\frac{1}{p}} \leq V(f).$$

Therefore, by replacing q by q/p in Lemma 2.3, Lemma 2.5 and Corollary 2.4, we get

$$\begin{aligned} \omega_q\left(\frac{1}{n}, f\right)^q &= \sup_{0 < h \leq \frac{1}{n}} \int_0^{\frac{1}{n}} \sum_{k=1}^n x_k^{\frac{q}{p}} dx \\ &\leq \int_0^{\frac{1}{n}} V^q(f) \max_{1 \leq k \leq n} \frac{k}{\left(\sum_{i=1}^k 1/\lambda_i\right)^{\frac{q}{p}}} dx \\ &= \frac{1}{n} V^q(f) \max_{1 \leq k \leq n} \frac{k}{\left(\sum_{i=1}^k 1/\lambda_i\right)^{\frac{q}{p}}}. \end{aligned}$$

Necessity. Suppose (2) doesn't hold, that is, there are sequences n_k and m_k such that

$$(4) \quad n_k \geq 2^{k+2},$$

$$(5) \quad m_k \leq n_k,$$

$$(6) \quad \omega\left(\frac{1}{n_k}\right) \left(\frac{n_k}{m_k}\right)^{\frac{1}{q}} \left(\sum_{i=1}^{m_k} \frac{1}{\lambda_i}\right)^{\frac{1}{p}} < \frac{1}{2^{4k}},$$

where

$$\max_{1 \leq \rho \leq n_k} \frac{\rho}{\left(\sum_{i=1}^{\rho} 1/\lambda_i\right)^{\frac{1}{p}}} = \frac{m_k}{\left(\sum_{i=1}^{m_k} 1/\lambda_i\right)^{\frac{1}{p}}}.$$

Denote

$$(7) \quad \Phi_k := \frac{1}{\sum_{i=1}^{m_k} 1/\lambda_i}.$$

Consider

$$g_k(y) := \begin{cases} 2^{-k} \Phi_k^{1/p}, & y \in [\frac{1}{2^k} + \frac{2j-2}{n_k}, \frac{1}{2^k} + \frac{2j-1}{n_k}); \quad 1 \leq j \leq N_k, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$(8) \quad s_k = \max\{j \in \mathbb{N} : 2j \leq \frac{n_k}{2^k} + 1\},$$

and

$$(9) \quad N_k = \min\{m_k, s_k\}.$$

Hence, applying the fact $2(s_k + 1) \geq \frac{n_k}{2^k} + 1$ and (4), we have

$$(10) \quad \frac{2s_k - 1}{n_k} \geq 2^{-k-1}.$$

The functions g_k have disjoint support. Thus $g := \sum_{k=1}^{\infty} g_k$ is a well-defined function on $[0, 1]$. Since

$$\begin{aligned} \|g\| &\leq \sum_{k=1}^{\infty} \|g_k\| \\ &= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{2N_k} \frac{|2^{-k} \Phi_k^{1/p}|^p}{\lambda_j} \right)^{1/p} \\ &\leq \sum_{k=1}^{\infty} 2^{-k} \left(2 \sum_{j=1}^{N_k} \frac{|\Phi_k|}{\lambda_j} \right)^{1/p} \\ &\stackrel{(9)}{\leq} \sum_{k=1}^{\infty} 2^{-k} \left(2 \sum_{j=1}^{m_k} \frac{|\Phi_k|}{\lambda_j} \right)^{1/p} \\ &\stackrel{(7)}{=} \sum_{k=1}^{\infty} 2^{-k} \left(2 \frac{\sum_{j=1}^{m_k} \frac{1}{\lambda_j}}{\sum_{j=1}^{m_k} \frac{1}{\lambda_j}} \right)^{1/p} \\ &< +\infty, \end{aligned}$$

we observe that $g \in \Lambda BV^{(p)}$.

If $N_k = m_k$, then

$$(11) \quad \frac{(2N_k - 1)}{n_k} \cdot |\Phi_k|^{\frac{q}{p}} \geq \frac{1}{\left(\frac{n_k}{m_k}\right) \left(\sum_{i=1}^{m_k} 1/\lambda_i\right)^{\frac{q}{p}}}$$

and if $N_k = s_k$ then

$$(12) \quad \frac{(2N_k - 1)}{n_k} \cdot |\Phi_k|^{\frac{q}{p}} \stackrel{(5),(10)}{\geq} 2^{-k-1} \cdot \frac{m_k}{n_k \left(\sum_{i=1}^{m_k} 1/\lambda_i\right)^{\frac{q}{p}}}.$$

Since $|g(x + \frac{1}{n_k}) - g(x)| = 2^{-k} \Phi_k^{1/p}$ for $x \in [\frac{1}{2^k}, \frac{1}{2^k} + \frac{2N_k-1}{n_k}]$, we get

$$\begin{aligned}
\omega_q(\frac{1}{n_k}, g)^q &= \sup_{0 < \gamma \leq \frac{1}{n_k}} \int_0^1 |g(x + \gamma) - g(x)|^q dx \\
&\geq \int_0^1 |g(x + \frac{1}{n_k}) - g(x)|^q dx \\
&\geq \int_{\frac{1}{2^k}}^{\frac{1}{2^k} + \frac{2N_k-1}{n_k}} |g(x + \frac{1}{n_k}) - g(x)|^q dx \\
&= \frac{2N_k-1}{n_k} \cdot 2^{-kq} |\Phi_k|^{\frac{q}{p}} \\
&\stackrel{(11),(12)}{\geq} \frac{1}{2^{k+qk+1}} \cdot \frac{m_k}{n_k (\sum_{i=1}^{m_k} 1/\lambda_i)^{\frac{q}{p}}},
\end{aligned}$$

and finally

$$\begin{aligned}
\frac{\omega_q(\frac{1}{n_k}, g)}{\omega(\frac{1}{n_k})} &\geq \left(\frac{1}{2^{k+qk+1}}\right)^{\frac{1}{q}} \cdot \frac{1}{\omega(\frac{1}{n_k})} \cdot \left(\frac{1}{\left(\frac{n_k}{m_k}\right)^{\frac{1}{q}} (\sum_{i=1}^{m_k} 1/\lambda_i)^{\frac{1}{p}}}\right) \\
&\stackrel{(6)}{\geq} 2^k \xrightarrow[k \rightarrow \infty]{} +\infty,
\end{aligned}$$

which shows that $g \notin H_\omega^q$. \square

Condition (2) simplifies when $p \geq q$. Thus we have the following Corollary.

Corollary 2.6. *For $p, q \in [1, \infty)$ ($p \geq q$), the inclusion $\Lambda BV^{(p)} \subset H_q^\omega$ holds if and only if*

$$(13) \quad \limsup_{n \rightarrow \infty} \left\{ \frac{1}{\omega(1/n) (\sum_{i=1}^n \frac{1}{\lambda_i})^{\frac{1}{p}}} \right\} < +\infty.$$

3. ACKNOWLEDGMENTS

The author is so grateful to Professor Hjalmar Rosengren for valuable comments, helpful discussions and for reviewing earlier drafts very carefully. The author also thanks Peter Hegarty for pointing out a mistake in an earlier version.

REFERENCES

- [1] W. W. Breckner, T. Trif, *On the singularities of certain families of nonlinear mappings*, Pure Math. Appl. 6 (1995) 121–137
- [2] W. W. Breckner, T. Trif, C. Varga, *Some applications of the condensation of the singularities of families of nonnegative functions*, Anal. Math. 25 (1999) 12–32
- [3] U. Goginava, *On the embedding of the Waterman class in the class H_p^ω* , Ukrainian Math. J. 57 (2005) 1818–1824
- [4] M. Hormozi, A. A. Ledari, F. Prus-Wisniowski, *On p - Λ -bounded variation*, Bull. Iranian Math. Soc. 37(4) (2011) 29–43
- [5] Y. E. Kuprikov, *Moduli of continuity of functions from Waterman classes*, Moscow Univ. Math. Bull. 52(5) (1997) 46–49
- [6] L. Leindler, *A note on embedding of classes H^ω* , Anal. Math. 27 (2001) 71–76
- [7] M. Schramm, D. Waterman, *On the magnitude of Fourier coefficients*, Proc. Amer. Math. Soc. 85 (1982) 407–410
- [8] M. Schramm, D. Waterman, *Absolute convergence of Fourier series of functions of $\Lambda BV^{(p)}$ and $\Phi \Lambda BV$* , Acta Math. Hungar. 40 (1982) 273–276
- [9] M. Shiba, *On the absolute convergence of Fourier series of functions of class $\Lambda BV^{(p)}$* , Sci. Rep. Fukushima Univ. 30 (1980) 7–10
- [10] R. G. Vyas, *On the absolute convergence of small gaps Fourier series of functions of $\Lambda BV^{(p)}$* , J. Inequal. Pure Appl. Math. 6(1) (2005) Article 23
- [11] R. G. Vyas, *On the convolution of functions of generalized bounded variation*, Georgian Math. J. 13 (2006) 193–197
- [12] R. G. Vyas, *Properties of functions of generalized bounded variation*, Mat. Vesnik 58 (2006) 91–96
- [13] H. Wang, *Embedding of Lipschitz classes into classes of functions of Λ -bounded variation*, J. Math. Anal. Appl. 354(2) (2009) 698–703
- [14] D. Waterman, *On convergence of Fourier series of functions of bounded generalized variation*, Studia Math. 44 (1972) 107–117
- [15] D. Waterman, *On Λ -bounded variation*, Studia Math. 57 (1976) 33–45

(Mahdi Hormozi) DEPARTMENT OF MATHEMATICAL SCIENCES, DIVISION OF MATHEMATICS, CHALMERS UNIVERSITY OF TECHNOLOGY AND UNIVERSITY OF GOTHENBURG, GOTHENBURG 41296, SWEDEN

E-mail address, Mahdi Hormozi: `hormozi@chalmers.se`